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BLO spaces associated with the Ornstein–Uhlenbeck operator

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Abstract

Let $(\mathbb{R}^n, |\cdot|, d\gamma)$ be the Gauss measure metric space, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $|\cdot|$ the Euclidean norm and $d\gamma(x) \equiv \pi^{-n/2}e^{-|x|^2}\,dx$ for all $x \in \mathbb{R}^n$ the Gauss measure. In this paper, for any $a \in (0, \infty)$, the authors introduce some $\mathrm{BLO}_a(\gamma)$ space, namely, the space of functions with bounded lower oscillation associated with a given class of admissible balls with parameter a. Then the authors prove that the noncentered local natural Hardy–Littlewood maximal operator is bounded from $\mathrm{BMO}(\gamma)$ of Mauceri and Meda to $\mathrm{BLO}_a(\gamma)$. Moreover, a characterization of the space $\mathrm{BLO}_a(\gamma)$, via the local natural maximal operator and $\mathrm{BMO}(\gamma)$, is given. The authors further prove that a class of maximal singular integrals, including the corresponding maximal operators of both imaginary powers of the Ornstein–Uhlenbeck operator and Riesz transforms of any order associated with the Ornstein–Uhlenbeck operator, are bounded from $L^\infty(\gamma)$ to $\mathrm{BLO}_a(\gamma)$.

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1. Introduction

Let $(\mathbb{R}^n, |\cdot|, d\gamma)$ be the Gauss measure metric space, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $|\cdot|$ the Euclidean norm and $d\gamma(x) \equiv \pi^{-n/2} e^{-|x|^2}\,dx$ for all $x \in \mathbb{R}^n$ the Gauss measure. Such an underlying space naturally appears in the analysis associated with the Ornstein–Uhlenbeck operator; see, for example, [5–7,11–14]. It is well known (see [11]) that $(\mathbb{R}^n, |\cdot|, d\gamma)$ is not a space of homogeneous type in the sense of Coifman and Weiss [4]. Thus, results for spaces of homogeneous type can not be trivially generalized to $(\mathbb{R}^n, |\cdot|, d\gamma)$ because of this non-doubling property of the Gauss measure.

Recently, via introducing the Hardy space $H^1(\gamma)$ and its dual space $\mathrm{BMO}(\gamma)$ related to a certain class $\mathcal{B}_a$ with $a \in (0,\infty)$ of admissible balls, Mauceri and Meda [11] developed a theory of singular integrals on $(\mathbb{R}^n, |\cdot|, d\gamma)$ which plays for the Ornstein–Uhlenbeck operator the same role as that the theory of classical Calderón–Zygmund operators plays for the Laplacian on classical Euclidean spaces. In other words, the space $\mathrm{BMO}(\gamma)$ is an appropriate space to study singular integrals associated with the Ornstein–Uhlenbeck operator; see [11] for more details. It was also pointed out by Mauceri and Meda [11, p. 280] that although the Gauss measure has the polynomial growth, the space $\mathrm{RBMO}(\gamma)$ of Tolsa [16] is not suitable for the study of singular integrals associated with the Ornstein–Uhlenbeck operator. We mention that the results of [11] are further generalized to certain locally doubling measure metric spaces in [2].

On the other hand, Coifman and Rochberg [3] introduced the $\mathrm{BLO}(\mathbb{R}^n)$ space on the classical Euclidean space, that is, the space of functions with bounded lower oscillation. Precisely, a locally integrable function f on $\mathbb{R}^n$ is said to be in $\mathrm{BLO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BLO}(\mathbb{R}^n)} \equiv \sup_Q \left[\frac{1}{|Q|}\int_Q f(x)\,dx - \operatorname*{essinf}_{y\in Q} f(y)\right] < \infty, \tag{1.1}$$

where Q denotes any cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ the Lebesgue measure of Q. Moreover, Bennett [1] obtained a characterization of $\mathrm{BLO}(\mathbb{R}^n)$ via the natural maximal operator and the classical $\mathrm{BMO}(\mathbb{R}^n)$ space and Leckband [10] proved the boundedness of certain maximal singular integrals from $L^\infty(\mathbb{R}^n) \cap L^{p_0}(\mathbb{R}^n)$ with certain $p_0 \in [1,\infty)$ to $\mathrm{BLO}(\mathbb{R}^n)$. Jiang [9] also introduced some BLO-type spaces for non-doubling measures with polynomial growth, which has a further extension in [8]. Because of the same reason as that pointed out in [11, p. 280], the theory of BLO spaces in [8,9], a fortiori, [1,3], can not be applied to singular integrals associated with the Ornstein–Uhlenbeck operator.

Let $a \in (0,\infty)$. In this paper, motivated by the space $\mathrm{BMO}(\gamma)$ of Mauceri and Meda [11], we introduce certain $\mathrm{BLO}_a(\gamma)$ spaces associated with the admissible balls $\mathcal{B}_a$, which seems to be suitable for singular integrals associated with the Ornstein–Uhlenbeck operator. The main difference of these BLO-type spaces between the current case and the classical case exists in that instead of taking the supremum over all cubes $Q \subset \mathbb{R}^n$ in (1.1), we only consider balls in $\mathcal{B}_a$; see Definition 2.1 below. Using the geometrical properties of Gauss measures, we then prove that the corresponding local natural maximal operator $\mathfrak{M}_a$ (see (2.6) below) is bounded from $\mathrm{BMO}(\gamma)$ to $\mathrm{BLO}_a(\gamma)$; see Theorem 3.1 below. As a consequence, we obtain that the non-centered local Hardy–Littlewood maximal operator $\mathcal{M}_a$ (see (2.2) below) is also bounded from $\mathrm{BMO}(\gamma)$ to $\mathrm{BLO}_a(\gamma)$; see Corollary 3.1 below. From Theorem 3.1, we then deduce a characterization of the space $\mathrm{BLO}_a(\gamma)$ via the local natural maximal operator and $\mathrm{BMO}(\gamma)$; see Theorem 3.2 below. In Section 4, we prove that a class of maximal singular integrals related to the parameter a (see (4.3) below), including the corresponding maximal operators of both imaginary powers of the Ornstein–Uhlenbeck operator and Riesz transforms of any order associated with the Ornstein–Uhlenbeck operator, are bounded from $L^\infty(\gamma)$ to $\mathrm{BLO}_a(\gamma)$; see Theorem 4.1 and Corollary 4.1 below.

We make some conventions on notation. Let $\mathbb{N} \equiv \{1, 2, \ldots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For any set $E \subset \mathbb{R}^n$, set $E^{\complement} \equiv \mathbb{R}^n \setminus E$. Denote by χ_E the characteristic function of any set $E \subset \mathbb{R}^n$. We also denote by C a positive constant independent of the main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ to denote $f \leqslant Cg$. If $f \lesssim g \lesssim f$, we then write $f \sim g$.

2. $\mathrm{BLO}_a(\gamma)$ and preliminaries

We begin with recalling some notation and notions; see, for example, [11]. Let m be a positive function on $\mathbb{R}^n$ defined by

$$m(x) \equiv \min\{1, 1/|x|\}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

For each ball $B \subset \mathbb{R}^n$, denote by c_B and r_B the center and radius of B, respectively. For any $\kappa > 0$, denote by κB the ball with center c_B and radius κr_B. For any $a \in (0, \infty)$, the admissible class $\mathcal{B}_a$ of balls is defined to be the set of all balls $B \subset \mathbb{R}^n$ such that $r_B \leqslant a m(c_B)$. For any $a > 0$ and $x \in \mathbb{R}^n$, denote by $\mathcal{B}_a(x)$ the collection of balls $B \in \mathcal{B}_a$ containing x.

For each positive number a and any locally integrable function f, the noncentered local Hardy–Littlewood maximal function $\mathcal{M}_a f$ is defined by

$$\mathcal{M}_a f(x) \equiv \sup_{B \in \mathcal{B}_a(x)} \frac{1}{\gamma(B)} \int_B |f(y)|\, d\gamma(y), \quad x \in \mathbb{R}^n; \tag{2.2}$$

see [11, (3.1)]. It is known that for any $a \in (0, \infty)$, the operator $\mathcal{M}_a$ is bounded on $L^p(\gamma)$ for every $p \in (1, \infty]$ and of weak type (1, 1); see [11, Theorem 3.1].

For any $a \in (0, \infty)$, $p \in [1, \infty)$ and any locally integrable function f on $\mathbb{R}^n$, set

$$\|f\|_*^{\mathcal{B}_a, p} \equiv \sup_{B \in \mathcal{B}_a} \left\{ \frac{1}{\gamma(B)} \int_B |f(x) - f_B|^p \, d\gamma(x) \right\}^{1/p}, \tag{2.3}$$

where and in what follows $f_B \equiv \frac{1}{\gamma(B)} \int_B f(y)\, d\gamma(y)$. By [11, Proposition 2.4] and repeating the discussion in [11, Section 4], we know that there exists a positive constant C, depending on $a \in (0, \infty)$ and $p \in [1, \infty)$, but not on f, such that

$$C^{-1} \|f\|_*^{\mathcal{B}_1, 1} \leqslant \|f\|_*^{\mathcal{B}_a, p} \leqslant C \|f\|_*^{\mathcal{B}_1, 1}. \tag{2.4}$$

For simplicity, we write $\|\cdot\|_*$ instead of $\|\cdot\|_*^{\mathcal{B}_1, 1}$. A function $f \in L^1(\gamma)$ is said to be in $\mathrm{BMO}(\gamma)$ if

$$\|f\|_{\mathrm{BMO}(\gamma)} \equiv \|f\|_* + \|f\|_{L^1(\gamma)} < \infty; \tag{2.5}$$

see [11, p. 281].

Correspondingly, when $a \in (0, \infty)$, the local natural Hardy–Littlewood maximal operator $\mathfrak{M}_a$ is defined by that for all locally integrable functions f on $\mathbb{R}^n$ and all $x \in \mathbb{R}^n$,

$$\mathfrak{M}_a f(x) \equiv \sup_{B \in \mathcal{B}_a(x)} \frac{1}{\gamma(B)} \int_B f(y)\, d\gamma(y). \tag{2.6}$$

Observe that $|\mathfrak{M}_a f(x)| \leqslant \mathcal{M}_a f(x)$ for all $a \in (0,\infty)$ and $x \in \mathbb{R}^n$.

Now, for $a \in (0,\infty)$, we introduce the following $\mathrm{BLO}_a(\gamma)$ space, namely, the space of locally integrable functions on $(\mathbb{R}^n, |\cdot|, d\gamma)$ with bounded lower oscillation.

Definition 2.1. Let $a \in (0,\infty)$. Then the space $\mathrm{BLO}_a(\gamma)$ is defined to be the set of all locally integrable functions f such that

$$\|f\|_{\mathrm{BLO}_a(\gamma)} \equiv \|f\|_{L^1(\gamma)} + \sup_{B\in\mathcal{B}_a}\left[\frac{1}{\gamma(B)}\int_B f(y)\,d\gamma(y) - \operatorname*{essinf}_{x\in B} f(x)\right] < \infty. \tag{2.7}$$

Remark 2.1. (i) For all $a \in (0,\infty)$ and $f \in L^\infty(\gamma)$, $\|f\|_{\mathrm{BLO}_a(\gamma)} \leqslant 3\|f\|_{L^\infty(\gamma)}$. Thus, $L^\infty(\gamma) \subset \mathrm{BLO}_a(\gamma)$.

(ii) If $a \in (0,\infty)$ and $f \in \mathrm{BLO}_a(\gamma)$, then $\|f\|_*^{\mathcal{B}_a,1} \leqslant 2\|f\|_{\mathrm{BLO}_a(\gamma)}$. In fact, for any $f \in \mathrm{BLO}_a(\gamma)$, write

$$\|f\|_*^{\mathcal{B}_a,1} = \sup_{B\in\mathcal{B}_a}\left\{\frac{1}{\gamma(B)}\int_{f(z)\geqslant f_B}\big[f(z)-f_B\big]\,d\gamma(z) + \frac{1}{\gamma(B)}\int_{f(z)<f_B}\big[f_B - f(z)\big]\,d\gamma(z)\right\}.$$

Notice that the first term in the above bracket is bounded by

$$\begin{aligned}\frac{1}{\gamma(B)}\int_{f(z)\geqslant f_B}\Big[f(z)-\operatorname*{essinf}_{y\in B} f(y)\Big]\,d\gamma(z) &\leqslant \frac{1}{\gamma(B)}\int_B\Big[f(z)-\operatorname*{essinf}_{y\in B} f(y)\Big]\,d\gamma(z)\\ &\leqslant \|f\|_{\mathrm{BLO}_a(\gamma)},\end{aligned}$$

while the second in the above bracket is bounded by

$$\frac{1}{\gamma(B)}\int_{f(z)<f_B}\Big[f_B-\operatorname*{essinf}_{y\in B} f(y)\Big]\,d\gamma(z) \leqslant f_B - \operatorname*{essinf}_{y\in B} f(y) \leqslant \|f\|_{\mathrm{BLO}_a(\gamma)}.$$

Thus $\|f\|_*^{\mathcal{B}_a,1} \leqslant 2\|f\|_{\mathrm{BLO}_a(\gamma)}$.

From this and (2.4), it follows that for any $a \in (0,\infty)$ and $p \in [1,\infty)$, there exists a positive constant C, depending only on a, p and n, such that for all $f \in \mathrm{BLO}_a(\gamma)$, $\|f\|_*^{\mathcal{B}_a,p} \leqslant C\|f\|_{\mathrm{BLO}_a(\gamma)}$. Thus, $\mathrm{BLO}_a(\gamma) \subset \mathrm{BMO}(\gamma)$.

(iii) We remark that all inclusions in $L^\infty(\gamma) \subset \mathrm{BLO}_a(\gamma) \subset \mathrm{BMO}(\gamma)$ for all $a \in (0,\infty)$ are proper; see Lemma 2.2 below.

Some geometry properties concerned with the Gauss measure are used throughout the whole paper. An important one, among others, is that the Gauss measure is indeed doubling on all balls in $\mathcal{B}_a$. To be precise, for all $\tau, a \in (0,\infty)$ and $B \in \mathcal{B}_a$, we denote by B_τ^* the union of all balls B' that intersect B such that $r_{B'} \leqslant \tau r_B$. Then it was proved in [11, Proposition 2.1] that

$$\sigma_{a,\tau}^* \equiv \sup_{B\in\mathcal{B}_a}\frac{\gamma(B_\tau^*)}{\gamma(B)} \leqslant (2\tau+1)^n e^{4a(\tau+1)+a^2}, \tag{2.8}$$

which is deduced from the property that for all $B \in \mathcal{B}_a$ and $x \in B$,

$$e^{-2a-a^2} \leqslant e^{|c_B|^2-|x|^2} \leqslant e^{2a}. \tag{2.9}$$

We also need the following properties.

Lemma 2.1. *Let $a, b \in (0, \infty)$. Then the following hold*:

(i) *if $B \in \mathcal{B}_a$ and $y \in B$, then $(a+1)^{-1}m(y) \leqslant m(c_B) \leqslant (a+1)m(y)$;*
(ii) *if $B \in \mathcal{B}_a$, $B' \in \mathcal{B}_b$ and $B \cap B' \neq \emptyset$, then*

$$(1+a+b)^{-1}m(c_{B'}) \leqslant m(c_B) \leqslant (1+a+b)m(c_{B'}).$$

Proof. To show (i), recall that it was proved in [11, (3.4)] that for all $y \in B$ and $B \in \mathcal{B}_1$, $m(y) \leqslant 2m(c_B)$. An argument similar to that also works for the general case. Precisely, for $B \in \mathcal{B}_a$ and $y \in B$, to show

$$m(y) \leqslant (a+1)m(c_B), \tag{2.10}$$

we first notice that if $|c_B| \leqslant 1$, then $m(y) \leqslant 1 = m(c_B)$ and thus (2.10) holds trivially. If $|c_B| > 1$ and $|y| \leqslant 1$, then the proof of (2.10) is reduced to proving $1 \leqslant (a+1)/|c_B|$, which follows from that

$$|c_B| \leqslant |c_B - y| + |y| < r_B + |y| \leqslant am(c_B) + 1 \leqslant a + 1.$$

If $|c_B| > 1$ and $|y| > 1$, then (2.10) is equivalent to that $1/|y| \leqslant (a+1)/|c_B|$, which follows from that

$$|c_B| \leqslant |c_B - y| + |y| < r_B + |y| \leqslant am(c_B) + |y| \leqslant a + |y| \leqslant (a+1)|y|.$$

Therefore (2.10) holds. The inverse inequality $m(c_B) \leqslant (a+1)m(y)$ follows from a symmetric argument. Thus (i) holds.

To prove (ii), by symmetry, it suffices to show

$$m(c_B) \leqslant (1+a+b)m(c_{B'}). \tag{2.11}$$

Observe that (2.11) is trivial if $|c_{B'}| \leqslant 1$. If $|c_{B'}| > 1$ and $|c_B| \leqslant 1$, then (2.11) is equivalent to that $1 \leqslant (1+a+b)/|c_{B'}|$, which follows from the fact

$$|c_{B'}| \leqslant |c_{B'} - c_B| + |c_B| \leqslant r_B + r_{B'} + |c_B| \leqslant am(c_B) + bm(c_{B'}) + |c_B| \leqslant 1 + a + b.$$

Here in the second step of the above inequality, we used the assumption $B \cap B' \neq \emptyset$. If $|c_{B'}| > 1$ and $|c_B| > 1$, then (2.11) is equivalent to that $1/|c_B| \leqslant (1+a+b)/|c_{B'}|$, which follows from the fact

$$|c_{B'}| \leqslant am(c_B) + bm(c_{B'}) + |c_B| \leqslant a + b + |c_B| \leqslant (1+a+b)|c_B|.$$

Hence, (ii) holds. This finishes the proof of Lemma 2.1. □

Finally we conclude this section with the following two examples that imply Remark 2.1(iii).

Lemma 2.2. *Let $a \in (0, \infty)$. Then the following hold*:

(i) *if $f(x) \equiv \log(\frac{1}{|x|})\chi_{(0,a)}(x)$ for all $x \in \mathbb{R}$, then $f \in \mathrm{BLO}_a(\gamma)$ and $f \notin L^\infty(\gamma)$;*
(ii) *if $g(x) \equiv \log|x|$ for all $x \in \mathbb{R}$, then $g \in \mathrm{BMO}(\gamma)$ and $g \notin \mathrm{BLO}_a(\gamma)$.*

Proof. To see (i), we notice that $f \notin L^\infty(\gamma)$ and $f \in L^1(\gamma)$. Thus, to show $f \in \mathrm{BLO}_a(\gamma)$, by (2.9), it suffices to prove that

$$\sup_{B \in \mathcal{B}_a} \frac{1}{\gamma(B)} \int_B \Big[f(x) - \operatorname*{essinf}_{y \in B} f(y)\Big]\, d\gamma(x) \sim \sup_{B \in \mathcal{B}_a} \frac{1}{|B|} \int_B \Big[f(x) - \operatorname*{essinf}_{y \in B} f(y)\Big]\, dx < \infty,$$

where $|B|$ denotes the Lebesgue measure of B. This follows from a simple computation, and we omit the details here.

Now we show (ii). Observe that g is a classical $\mathrm{BMO}(\mathbb{R}^n)$ function; see, for example, [15, p. 140]. Therefore, by (2.9), we have

$$\sup_{B\in\mathcal{B}_a}\frac{1}{\gamma(B)}\int_B\left|g(z)-g_B\right|d\gamma(x)\leqslant\sup_{B\in\mathcal{B}_a}\frac{2}{\gamma(B)}\int_B\left|g(z)-\frac{1}{|B|}\int_B g(y)\,dy\right|d\gamma(x)$$
$$\sim\sup_{B\in\mathcal{B}_a}\frac{1}{|B|}\int_B\left|g(z)-\frac{1}{|B|}\int_B g(y)\,dy\right|dx<\infty,$$

which together with the fact that $g\in L^1(\gamma)$ implies $g\in\mathrm{BMO}(\gamma)$. To demonstrate that $g\notin\mathrm{BLO}_a(\gamma)$, we consider the intervals $B_n\equiv(1/n,1/n+1/\sqrt{n})$ for large $n\in\mathbb{N}$. It is easy to show that $B_n\in\mathcal{B}_a$ and $B_n\subset(0,a)$ for n large enough. Again, using (2.9), we obtain

$$\frac{1}{\gamma(B_n)}\int_{B_n}\Big[g(x)-\operatorname*{essinf}_{y\in B_n}g(y)\Big]\,d\gamma(x)\sim\frac{1}{|B_n|}\int_{B_n}\Big[g(x)-\operatorname*{essinf}_{y\in B}g(y)\Big]\,dx$$
$$=\frac{1}{|B_n|}\int_{1/n}^{1/n+1/\sqrt{n}}\big[\log x-\log(1/n)\big]\,dx$$
$$=\frac{1+\sqrt{n}}{\sqrt{n}}\log(1+\sqrt{n})-1,$$

which tends to infinity as $n\to\infty$. Therefore, $g\notin\mathrm{BLO}_a(\gamma)$. This finishes the proof of Lemma 2.2. □

3. Local Hardy–Littlewood maximal operators

One of the main results in this section is that the operator $\mathfrak{M}_a$ for $a\in(0,\infty)$ as in (2.6) is bounded from $\mathrm{BMO}(\gamma)$ to $\mathrm{BLO}_a(\gamma)$.

Theorem 3.1. *Let $a\in(0,\infty)$. Then there exists a positive constant C, depending on a and n, such that for all $B\in\mathcal{B}_a$ and all locally integrable functions f satisfying that $\|f\|_*<\infty$,*

$$\frac{1}{\gamma(B)}\int_B\mathfrak{M}_a f(y)\,d\gamma(y)\leqslant C\|f\|_*+\inf_{x\in B}\mathfrak{M}_a f(x);\tag{3.1}$$

moreover, for all $f\in\mathrm{BMO}(\gamma)$, $\|\mathfrak{M}_a f\|_{\mathrm{BLO}_a(\gamma)}\leqslant C\|f\|_{\mathrm{BMO}(\gamma)}$.

Proof. To show (3.1), fix any $B\in\mathcal{B}_a$. Write f as

$$f=[f-f_B]\chi_{3B}+[f_B\chi_{3B}+f\chi_{(3B)^\complement}].$$

By the fact $\mathfrak{M}_a f\leqslant\mathcal{M}_a f$, Hölder's inequality and the $L^2(\gamma)$-boundedness of $\mathcal{M}_a$, we obtain

$$\frac{1}{\gamma(B)}\int\limits_B \mathfrak{M}_a\big([f-f_B]\chi_{3B}\big)(y)\,d\gamma(y)$$
$$\leqslant \left\{\frac{1}{\gamma(B)}\int\limits_B \big|\mathcal{M}_a\big([f-f_B]\chi_{3B}\big)(y)\big|^2\,d\gamma(y)\right\}^{1/2}$$
$$\leqslant \|\mathcal{M}_a\|_{L^2(\gamma)\to L^2(\gamma)}\left\{\frac{1}{\gamma(B)}\int\limits_{3B} \big|f(y)-f_B\big|^2\,d\gamma(y)\right\}^{1/2},$$

where $\|\mathcal{M}_a\|_{L^2(\gamma)\to L^2(\gamma)}$ denotes the operator norm of $\mathcal{M}_a$ in $L^2(\gamma)$. Using Minkowski's inequality and (2.8), we then have

$$\left\{\frac{1}{\gamma(B)}\int\limits_{3B} \big|f(y)-f_B\big|^2\,d\gamma(y)\right\}^{1/2}$$
$$\leqslant \left\{\frac{1}{\gamma(B)}\int\limits_{3B} \big|f(y)-f_{3B}\big|^2\,d\gamma(y)\right\}^{1/2}+|f_{3B}-f_B|$$
$$\leqslant \left(\frac{\gamma(3B)}{\gamma(B)}\right)^{1/2}\|f\|_*^{\mathcal{B}_{3a},2}+\frac{1}{\gamma(B)}\int\limits_B \big|f(y)-f_{3B}\big|\,d\gamma(y)$$
$$\leqslant \big(\sigma_{a,3}^*\big)^{1/2}\|f\|_*^{\mathcal{B}_{3a},2}+\sigma_{a,3}^*\|f\|_*^{\mathcal{B}_{3a},1},$$

which together with (2.4) further yields

$$\frac{1}{\gamma(B)}\int\limits_B \mathfrak{M}_a\big([f-f_B]\chi_{3B}\big)(y)\,d\gamma(y)\leqslant C\|f\|_*,$$

where C is a positive constant depending only on a and n. Therefore, the proof of (3.1) is reduced to proving that

$$\frac{1}{\gamma(B)}\int\limits_B \mathfrak{M}_a(f_B\chi_{3B}+f\chi_{(3B)^\complement})(y)\,d\gamma(y)\leqslant C\|f\|_*+\inf_{x\in B}\mathfrak{M}_a(f)(x). \tag{3.2}$$

To obtain (3.2), it suffices to show that for all $y\in B$,

$$\mathfrak{M}_a(f_B\chi_{3B}+f\chi_{(3B)^\complement})(y)\leqslant C\|f\|_*+\inf_{x\in B}\mathfrak{M}_a(f)(x), \tag{3.3}$$

which can be deduced from that for all balls $B'\in\mathcal{B}_a(y)$ and all $x\in B$,

$$\frac{1}{\gamma(B')}\int\limits_{B'} \big[f_B\chi_{3B}(z)+f(z)\chi_{(3B)^\complement}(z)\big]\,d\gamma(z)\leqslant C\|f\|_*+\mathfrak{M}_a(f)(x), \tag{3.4}$$

where C is a positive constant depending only on a and n.

The estimate (3.4) is trivial if $\mathfrak{M}_a(f)(x)=\infty$. Now assume that $\mathfrak{M}_a(f)(x)<\infty$. In this case, from the local integrability of f and the fact that $f_B\leqslant\mathfrak{M}_a(f)(x)$, it follows that $\mathfrak{M}_a(f)(x)>-\infty$. We now prove (3.4) by considering the subcases $B'\subset(3B)$ and $B'\cap(3B)^\complement\neq\emptyset$, respectively. If $B'\subset(3B)$, then the left-hand side of (3.4) equals

$$\frac{1}{\gamma(B')}\int\limits_{B'} f_B\,d\gamma(z)=f_B\leqslant\mathfrak{M}_a f(x)\leqslant C\|f\|_*+\mathfrak{M}_a(f)(x).$$

It remains to show (3.4) in the case $B' \cap (3B)^{\complement} \neq \emptyset$. Notice that $y \in B' \cap B$ and $B' \cap (3B)^{\complement} \neq \emptyset$ imply that

$$r_B < r_{B'}. \tag{3.5}$$

Indeed, if $r_{B'} \leqslant r_B$, then for all $z \in B'$, we have $|z - c_B| \leqslant |z - y| + |y - c_B| < 3r_B$ and hence $B' \subset (3B)$. This contradiction implies (3.5).

Set $\nu \equiv 3(1+2a)$. Assume that $r_{B'} \leqslant \nu r_B$. Using the hypothesis $B' \cap (3B)^{\complement} \neq \emptyset$ and the fact that $f_B \leqslant \mathfrak{M}_a f(x)$, we obtain

$$\begin{aligned}
&\frac{1}{\gamma(B')} \int_{B'} \big[f_B \chi_{3B}(z) + f(z)\chi_{(3B)^{\complement}}(z)\big]\, d\gamma(z) - \mathfrak{M}_a(f)(x) \\
&\quad = \frac{1}{\gamma(B')} \int_{B'} \big[\big(f(z) - f_B\big)\chi_{(3B)^{\complement}}(z) + f_B\big]\, d\gamma(z) - \mathfrak{M}_a(f)(x) \\
&\quad \leqslant \frac{1}{\gamma(B')} \int_{B'} \big[f(z) - f_B\big]\chi_{(3B)^{\complement}}(z)\, d\gamma(z) \equiv \mathrm{J}.
\end{aligned} \tag{3.6}$$

Let $\widetilde{B} \equiv B(c_{B'}, 3r_{B'})$. By (3.5), it is easy to prove that $\widetilde{B}$ contains both B and B'. Moreover, by (2.8) and the assumption $r_{B'} \leqslant \nu r_B$, we have $\gamma(\widetilde{B}) \leqslant \sigma^*_{a,3}\gamma(B') \leqslant \sigma^*_{a,3}\sigma^*_{a,\nu}\gamma(B)$. From this and the fact $\widetilde{B} \in \mathcal{B}_{3a}$, it follows that

$$\begin{aligned}
\mathrm{J} &\leqslant \frac{1}{\gamma(B')} \int_{B'} \big|f(z) - f_{\widetilde{B}}\big|\, d\gamma(z) + |f_{\widetilde{B}} - f_B| \\
&\leqslant \left[\frac{\gamma(\widetilde{B})}{\gamma(B')} + \frac{\gamma(\widetilde{B})}{\gamma(B)}\right] \|f\|_*^{\mathcal{B}_{3a},1} \leqslant \big[\sigma^*_{a,3}\big(1 + \sigma^*_{a,\nu}\big)\big] \|f\|_*^{\mathcal{B}_{3a},1},
\end{aligned}$$

which together with (3.6) yields (3.4) in the case $B' \cap (3B)^{\complement} \neq \emptyset$ and $r_{B'} \leqslant \nu r_B$.

Assume now that $r_{B'} > \nu r_B$. Set $\widetilde{D} \equiv B(c_B, 3r_{B'}/\nu)$. Observe that $3B \subset \widetilde{D}$. Write

$$\begin{aligned}
&\frac{1}{\gamma(B')} \int_{B'} \big[f_B \chi_{3B}(z) + f(z)\chi_{(3B)^{\complement}}(z)\big]\, d\gamma(z) - \mathfrak{M}_a(f)(x) \\
&\quad = \frac{1}{\gamma(B')} \int_{B'} \big[(f_B - f_{\widetilde{D}})\chi_{3B}(z)\big]\, d\gamma(z) \\
&\qquad + \frac{1}{\gamma(B')} \int_{B'} \big[\big(f(z) - f_{\widetilde{D}}\big)\chi_{(3B)^{\complement}}(z)\big]\, d\gamma(z) + \big[f_{\widetilde{D}} - \mathfrak{M}_a(f)(x)\big] \\
&\quad \equiv \mathrm{Z}_1 + \mathrm{Z}_2 + \mathrm{Z}_3.
\end{aligned} \tag{3.7}$$

By $B' \in \mathcal{B}_a$, $B \in \mathcal{B}_a$, $B \cap B' \neq \emptyset$, $\nu \equiv 3(1+2a)$ and Lemma 2.1(ii), we obtain

$$3r_{B'}/\nu \leqslant am(c_{B'})/(1+2a) \leqslant am(c_B),$$

and hence, $\widetilde{D} \in \mathcal{B}_a$. This combined with the fact $x \in B \subset \widetilde{D}$ implies $\mathrm{Z}_3 \leqslant 0$.

To estimate Z_1, notice that $y \in (B \cap B') \subset (\widetilde{D} \cap B')$ and $r_{\widetilde{D}} = 3r_{B'}/\nu$ together with (2.8) imply that $\gamma(\widetilde{D}) \leqslant \sigma^*_{a,3/\nu}\gamma(B')$. Moreover, $\gamma(3B) \leqslant \sigma^*_{a,3}\gamma(B)$. Thus,

$$\mathrm{Z}_1 \leqslant \frac{\gamma(B'\cap(3B))}{\gamma(B')}\frac{1}{\gamma(B)}\int_B \left|f(z)-f_{\widetilde{D}}\right|d\gamma(z)$$
$$\leqslant \frac{\gamma(3B)}{\gamma(B)}\frac{\gamma(\widetilde{D})}{\gamma(B')}\frac{1}{\gamma(\widetilde{D})}\int_{\widetilde{D}} \left|f(z)-f_{\widetilde{D}}\right|d\gamma(z) \leqslant \sigma^*_{a,3}\sigma^*_{a,3/\nu}\|f\|_*^{\mathcal{B}_a,1}.$$

To estimate Z_2, notice that

$$\mathrm{Z}_2 \leqslant \frac{1}{\gamma(B')}\int_{B'} \left|f(z)-f_{\widetilde{D}}\right|d\gamma(z)$$
$$\leqslant \frac{1}{\gamma(B')}\int_{B'} \left|f(z)-f_{B'}\right|d\gamma(z) + |f_{B'}-f_{\widetilde{D}}| \leqslant \|f\|_*^{\mathcal{B}_a,1} + |f_{B'}-f_{\widetilde{D}}|. \tag{3.8}$$

For any $z\in\widetilde{D}$, by (3.5) and $y\in(B\cap B')$, we have

$$|z-c_{B'}| \leqslant |z-c_B|+|c_B-y|+|y-c_{B'}| < 3r_{B'}/\nu + r_B + r_{B'} < (2+3/\nu)r_{B'} \leqslant 3r_{B'},$$

which implies that $\widetilde{D}\subset 3B'$. Therefore,

$$|f_{B'}-f_{\widetilde{D}}| \leqslant |f_{B'}-f_{3B'}|+|f_{3B'}-f_{\widetilde{D}}|$$
$$\leqslant \frac{1}{\gamma(B')}\int_{B'}\left|f(z)-f_{3B'}\right|d\gamma(z) + \frac{1}{\gamma(\widetilde{D})}\int_{\widetilde{D}}\left|f(z)-f_{3B'}\right|d\gamma(z)$$
$$\leqslant \left[\frac{\gamma(3B')}{\gamma(B')}+\frac{\gamma(3B')}{\gamma(\widetilde{D})}\right]\frac{1}{\gamma(3B')}\int_{\widetilde{D}}\left|f(z)-f_{3B'}\right|d\gamma(z)$$
$$\leqslant \left[\sigma^*_{a,3}+\sigma^*_{a,\nu}\right]\|f\|_*^{\mathcal{B}_{3a},1}.$$

This together with (3.8) further implies that $\mathrm{Z}_2 \lesssim \|f\|_*$.

Combining (3.7) and the estimates of Z_1 through Z_3 yields that (3.4) holds in the case $B'\cap(3B)^{\complement}\neq\emptyset$ and $r_{B'}>\nu r_B$. Thus, (3.1) holds.

To prove that $\mathfrak{M}_a$ is bounded from $\mathrm{BMO}(\gamma)$ to $\mathrm{BLO}_a(\gamma)$, we first notice that by (3.1),

$$\frac{1}{\gamma(B)}\int_B \mathfrak{M}_a f(x)\,d\gamma(x) \leqslant C\|f\|_* + \operatorname*{essinf}_{x\in B}\mathfrak{M}_a f(x). \tag{3.9}$$

Moreover, if $f\in\mathrm{BMO}(\gamma)$, then by the fact that $|\mathfrak{M}_a(f)|\leqslant\mathcal{M}_a(f)$ and that $\mathcal{M}_a$ is bounded from $L^1(\gamma)$ to weak-$L^1(\gamma)$, we obtain that for all $\lambda>0$,

$$\gamma\left(\left\{x\in\mathbb{R}^n\colon \left|\mathfrak{M}_a(f)(x)\right|>\lambda\right\}\right) \lesssim \|f\|_{L^1(\gamma)}/\lambda \lesssim \|f\|_{\mathrm{BMO}(\gamma)}/\lambda.$$

Then by letting $\lambda\to\infty$, we obtain that $|\mathfrak{M}_a(f)(x)|$ is finite for almost every $x\in\mathbb{R}^n$, which implies that $\operatorname{essinf}_{x\in B}\mathfrak{M}_a f(x)<\infty$ for all $B\in\mathcal{B}_a$. Recall that the differentiation theorem for the integral holds in the setting $(\mathbb{R}^n,|\cdot|,d\gamma)$; see the proof of Theorem 3.5 in [11]. Thus $f(x)\leqslant\mathfrak{M}_a f(x)$ for almost every $x\in\mathbb{R}^n$. From this and (3.9), it follows that if $\operatorname{essinf}_{x\in B}\mathfrak{M}_a f(x)=-\infty$ for some $B\in\mathcal{B}_a$, then

$$\frac{1}{\gamma(B)}\int_B f(x)\,d\gamma(x) \leqslant \frac{1}{\gamma(B)}\int_B \mathfrak{M}_a f(x)\,d\gamma(x) \leqslant C\|f\|_* + \operatorname*{essinf}_{x\in B}\mathfrak{M}_a(f)(x) = -\infty,$$

which contradicts to the local integrability of f. Thus, $\operatorname{essinf}_{x\in B}\mathfrak{M}_a f(x)$ is finite for all $B\in\mathcal{B}_a$. Subtracting $\operatorname{essinf}_{x\in B}\mathfrak{M}_a f(x)$ from both sides of (3.9) and using Definition 2.1 yield $\|f\|_{\mathrm{BLO}_a(\gamma)}\lesssim\|f\|_{\mathrm{BMO}(\gamma)}$. This concludes the proof of Theorem 3.1. □

Applying Theorem 3.1, we can easily deduce the boundedness of $\mathcal{M}_a$ from $\mathrm{BMO}(\gamma)$ to $\mathrm{BLO}_a(\gamma)$ as follows.

Corollary 3.1. *Let $a\in(0,\infty)$. Then $\mathcal{M}_a$ in* (2.2) *is bounded from* $\mathrm{BMO}(\gamma)$ *to* $\mathrm{BLO}_a(\gamma)$.

Proof. Observe that for all $f\in\mathrm{BMO}(\gamma)$ and $x\in\mathbb{R}^n$, $\mathcal{M}_a f(x)=\mathfrak{M}_a(|f|)(x)$ and $\||f|\|_{\mathrm{BMO}(\gamma)}\leqslant 2\|f\|_{\mathrm{BMO}(\gamma)}$, which together with Theorem 3.1 implies that for all $f\in\mathrm{BMO}(\gamma)$,

$$\|\mathcal{M}_a f\|_{\mathrm{BLO}_a(\gamma)}=\big\|\mathfrak{M}_a\big(|f|\big)\big\|_{\mathrm{BLO}_a(\gamma)}\lesssim\|f\|_{\mathrm{BMO}(\gamma)}.$$

This finishes the proof of Corollary 3.1. □

The following proposition reveals the role of the local natural maximal operator in characterizing the space $\mathrm{BLO}_a(\gamma)$.

Proposition 3.1. *Let $a\in(0,\infty)$. Then $f\in\mathrm{BLO}_a(\gamma)$ if and only if $f\in L^1(\gamma)$ and $\mathfrak{M}_a f-f\in L^\infty(\gamma)$. Moreover,*

$$\|\mathfrak{M}_a f-f\|_{L^\infty(\gamma)}=\sup_{B\in\mathcal{B}_a}\left[\frac{1}{\gamma(B)}\int\limits_B f(y)\,d\gamma(y)-\operatorname*{essinf}_{x\in B}f(x)\right].$$

Proof. Using the fact that $f(x)\leqslant\mathfrak{M}_a f(x)$ for all locally integrable functions on $\mathbb{R}^n$ and almost every $x\in\mathbb{R}^n$, and following the line of the proof of [1, Lemma 2], we then obtain Proposition 3.1. We omit the details here. □

From this proposition and Theorem 3.1, we deduce the following characterization of the space $\mathrm{BLO}_a(\gamma)$ via the space $\mathrm{BMO}(\gamma)$ and the local natural maximal operator $\mathfrak{M}_a$; see also [1] for the Euclidean case and [8] for the non-doubling measure case.

Theorem 3.2. *Let $a\in(0,\infty)$. Then $f\in\mathrm{BLO}_a(\gamma)$ if and only if there exist $h\in L^\infty(\gamma)$ and $F\in\mathrm{BMO}(\gamma)$ such that*

$$f=\mathfrak{M}_a F+h. \tag{3.10}$$

Moreover, there exists a positive constant C, depending only on a and n, such that

$$\|f\|_{\mathrm{BLO}_a(\gamma)}\sim\inf\big\{\|F\|_{\mathrm{BMO}(\gamma)}+\|f\|_{L^\infty(\gamma)}\big\},$$

where the infimum is taken over all the decompositions as in (3.10).

Proof. If f has a representation as in (3.10) and $\|F\|_{\mathrm{BMO}(\gamma)}<\infty$, then by Theorem 3.1, we obtain that $\mathfrak{M}_a F\in\mathrm{BLO}_a(\gamma)$ and $\|\mathfrak{M}_a F\|_{\mathrm{BLO}_a(\gamma)}\lesssim\|F\|_{\mathrm{BMO}(\gamma)}$. Notice that $h\in L^\infty(\gamma)\subset\mathrm{BLO}_a(\gamma)$ and $\|h\|_{\mathrm{BLO}_a(\gamma)}\leqslant 3\|h\|_{L^\infty(\gamma)}$ by Remark 2.1(i). Using the definition of $\mathrm{BLO}_a(\gamma)$, we are easy to see that $f\in\mathrm{BLO}_a(\gamma)$ and

$$\|f\|_{\mathrm{BLO}_a(\gamma)}\leqslant\|\mathfrak{M}_a F\|_{\mathrm{BLO}_a(\gamma)}+\|h\|_{\mathrm{BLO}_a(\gamma)}\lesssim\|F\|_{\mathrm{BMO}(\gamma)}+\|h\|_{L^\infty(\gamma)}.$$

Taking over all the decompositions as in (3.10) yields

$$\|f\|_{\mathrm{BLO}_a(\gamma)} \lesssim \inf\bigl\{\|F\|_{\mathrm{BMO}(\gamma)} + \|f\|_{L^\infty(\gamma)}\bigr\}.$$

Conversely, suppose that $f \in \mathrm{BLO}_a(\gamma)$. Notice that Proposition 3.1 implies that $f - \mathfrak{M}_a f \in L^\infty(\gamma)$ and also $\mathfrak{M}_a f$ is finite almost everywhere. Thus (3.10) holds with $F \equiv f$ and $h \equiv f - \mathfrak{M}_a f$. Moreover, by Remark 2.1(ii) and Proposition 3.1 again,

$$\|F\|_{\mathrm{BMO}(\gamma)} + \|h\|_{L^\infty(\gamma)} = \|f\|_{\mathrm{BMO}(\gamma)} + \|f - \mathfrak{M}_a f\|_{L^\infty(\gamma)} \lesssim \|f\|_{\mathrm{BLO}_a(\gamma)}.$$

This gives the desired estimate, which completes the proof of Theorem 3.2. □

4. Maximal singular integrals

In this section, we consider the boundedness from $L^\infty(\gamma)$ to $\mathrm{BLO}_a(\gamma)$ of certain maximal singular integrals associated with the Ornstein–Uhlenbeck operator. It is well known that the operator $\mathcal{L}_0 \equiv -\frac{1}{2}\triangle + x \cdot \nabla$, which has domain $C_c^\infty(\mathbb{R}^n)$, is essentially self-adjoint on $L^2(\gamma)$ and its closure $\mathcal{L}$ is called the Ornstein–Uhlenbeck operator. Moreover, $\mathcal{L}$ has the spectral resolution that $\mathcal{L}f = \sum_{k=0}^\infty k\mathcal{P}_k f$ for all $f \in \mathrm{Dom}(\mathcal{L})$, where $\mathcal{P}_k$ is the orthogonal projection onto the linear span of Hermite polynomials of degree k in n variables; see, for example, [11,14]. Assume that $M : \mathbb{Z}_+ \to \mathbb{C}$ is a bounded sequence. The spectral operator associated to the spectral multiplier M, denoted by $M(\mathcal{L})$, is defined by that $M(\mathcal{L})f \equiv \sum_{k=0}^\infty M(k)\mathcal{P}_k f$ for all $f \in L^2(\gamma)$.

Particularly, for any $u \in \mathbb{R}$, let $M_u : \mathbb{Z}_+ \to \mathbb{C}$ be defined by that $M_u(0) = 0$ and $M_u(k) = k^{iu}$ for $k \in \mathbb{N}$. Then the family of operators $\{M_u(\mathcal{L})\}_{u\in\mathbb{R}}$ are referred to as imaginary powers of the Ornstein–Uhlenbeck operator. It is known that these operators are all bounded on $L^p(\gamma)$ for $p \in (1, \infty)$, from $L^1(\gamma)$ to weak-$L^1(\gamma)$, from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\mathrm{BMO}(\gamma)$; see [5,6,11] and the references therein.

Another example of singular integrals associated to the Ornstein–Uhlenbeck operator is the Riesz transform. Precisely, for $b \in (0, \infty)$, let $P_b : \mathbb{Z}_+ \to \mathbb{C}$ be defined by that $P_b(0) = 0$ and $P_b(k) = k^{-b}$ for $k \in \mathbb{N}$. The operator $D^\alpha P_b(\mathcal{L})$ with $|\alpha| = 2b$ is called the Riesz transform of order α, where α is a multiindex and D^α denotes the partial derivative of order α. It is known that Riesz transforms are bounded on $L^p(\gamma)$ for all $p \in (1, \infty)$ and bounded from $L^\infty(\gamma)$ to $\mathrm{BMO}(\gamma)$, moreover, $D^\alpha P_b(\mathcal{L})$ is of weak type $(1, 1)$ if and only if $|\alpha| \leqslant 2$; see [11–14] and the references therein for the details.

Let T be a linear operator bounded on $L^2(\gamma)$ with kernel that coincides with a function K on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x): x \in \mathbb{R}^n\}$. That is, for any function $\phi \in C_c^\infty(\mathbb{R}^n)$ and all $x \notin \operatorname{supp}\phi$,

$$T(\phi)(x) = \int_{\mathbb{R}^n} K(x, y)\phi(y)\,d\gamma(y). \tag{4.1}$$

Further assume that

$$v_1 \equiv \sup_{B\in\mathcal{B}_1} \sup_{x,x'\in B} \int_{(2B)^\complement} \bigl|K(x, y) - K(x', y)\bigr|\,d\gamma(y) < \infty. \tag{4.2}$$

It was proved in [11] that such operators are bounded from $L^\infty(\gamma)$ to $\mathrm{BMO}(\gamma)$; moreover, typical examples of such operators are $\{M_u(\mathcal{L})\}_{u\in\mathbb{R}}$ and $D^\alpha P_b(\mathcal{L})$ with $|\alpha| = 2b > 0$.

For any T as above, $a \in (0, \infty)$ and all $x \in \mathbb{R}^n$, set

$$T_a^*(f)(x) \equiv \sup_{0<\epsilon\leqslant am(x)} \bigl|T_\epsilon(f)(x)\bigr|, \tag{4.3}$$

where m is as in (2.1) and for all $x \in \mathbb{R}^n$,

$$T_\epsilon(f)(x) \equiv \int_{|x-y|>\epsilon} K(x,y) f(y)\, d\gamma(y).$$

We obtain the boundedness of T_a^* from $L^\infty(\gamma)$ to $\mathrm{BLO}_a(\gamma)$ as follows.

Theorem 4.1. *Let $a \in (0,\infty)$. Suppose that T is a linear operator bounded on $L^2(\gamma)$ with kernel that coincides to a function K on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\colon x \in \mathbb{R}^n\}$ as in* (4.1)*. Moreover, assume that there exist positive constants C_a and v_a such that for all $|x-y| \leqslant 3a(a+1)^2 m(x)$ and $x \neq y$,*

$$\bigl|K(x,y)\bigr| \leqslant C_a e^{|y|^2} \frac{1}{|x-y|^n}, \tag{4.4}$$

and

$$\sup_{B \in \mathcal{B}_a} \sup_{x,x' \in B} \int_{(2B)^{\complement}} \bigl|K(x,y) - K(x',y)\bigr|\, d\gamma(y) \leqslant v_a. \tag{4.5}$$

Then the operator T_a^* as in (4.3) *is bounded from $L^\infty(\gamma)$ to* $\mathrm{BLO}_a(\gamma)$.

As an application of Theorem 4.1, we obtain the following boundedness of maximal singular integrals of imaginary powers and Riesz transforms of the Ornstein–Uhlenbeck operator from $L^\infty(\gamma)$ to $\mathrm{BLO}(\gamma)$.

Corollary 4.1. *Let $u \in \mathbb{R}$, $a, b \in (0,\infty)$ and α be a multiindex such that $|\alpha| = 2b$. Then the maximal singular integrals $(M_u(\mathcal{L}))_a^*$ and $(D^\alpha P_b(\mathcal{L}))_a^*$, defined as in* (4.3) *with T replaced, respectively, by $M_u(\mathcal{L})$ and $D^\alpha P_b(\mathcal{L})$, are bounded from $L^\infty(\gamma)$ to* $\mathrm{BLO}_a(\gamma)$.

Proof. From the proof of [11, Theorem 7.2], we deduce that the kernels of $M_u(\mathcal{L})$ and $D^\alpha P_b(\mathcal{L})$ satisfy (4.5). Moreover, it follows from [6, Lemma 2.1] and [12, Lemma 3.1], respectively, that the kernels of $M_u(\mathcal{L})$ and $D^\alpha P_b(\mathcal{L})$ satisfy (4.4). Therefore, the desired conclusions follow from Theorem 4.1, which completes the proof of Corollary 4.1. $\square$

The rest of this section is devoted to the proof of Theorem 4.1. To this end, we need certain kind of "Cotlar's inequality"; see, for example, [15, p. 34] for the Euclidean case.

Lemma 4.1. *Let all the notation be as in Theorem* 4.1*. Then for any given $\nu \in (0,1]$, there exists a positive constant C, depending on a, n and ν, such that for all $f \in L^\infty(\gamma)$ and all $x \in \mathbb{R}^n$,*

$$T_a^*(f)(x) \leqslant C\Bigl(\bigl[\mathcal{M}_a\bigl(\bigl[T(f)\bigr]^\nu\bigr)(x)\bigr]^{1/\nu} + \|f\|_{L^\infty(\gamma)}\Bigr).$$

Proof. It suffices to show that for all $x \in \mathbb{R}^n$ and all $\epsilon \in (0, am(x)]$,

$$\bigl|T_\epsilon(f)(x)\bigr| \lesssim \bigl[\mathcal{M}_a\bigl(\bigl[T(f)\bigr]^\nu\bigr)(x)\bigr]^{1/\nu} + \|f\|_{L^\infty(\gamma)}. \tag{4.6}$$

To obtain (4.6), fix any $x \in \mathbb{R}^n$ and $\epsilon \in (0, am(x)]$. Let $B \equiv B(x, \epsilon/2)$. Define $f_1 \equiv f\chi_{2B}$ and $f_2 \equiv f - f_1$. Then we have

$$T_\epsilon(f)(x) = \int_{|y-x|>\epsilon} K(x,y) f(y)\, d\gamma(y) = T(f_2)(x). \tag{4.7}$$

Notice that $B \in \mathcal{B}_a$. Thus, for all $z \in B$, by (4.5),

$$\left|T(f_2)(z) - T(f_2)(x)\right| \leqslant \int\limits_{(2B)^{\complement}} \left|K(z,w) - K(x,w)\right|\left|f(w)\right| d\gamma(w) \leqslant v_a \|f\|_{L^\infty(\gamma)}.$$

This combined with (4.7) yields that

$$\begin{aligned}\left|T_\epsilon(f)(x)\right| = \left|T(f_2)(x)\right| &\leqslant \left|T(f_2)(x) - T(f_2)(z)\right| + \left|T(f)(z)\right| + \left|T(f_1)(z)\right| \\ &\leqslant v_a \|f\|_{L^\infty(\gamma)} + \left|T(f)(z)\right| + \left|T(f_1)(z)\right|. \end{aligned} \tag{4.8}$$

If $T_\epsilon(f)(x) = 0$, then (4.6) trivially holds. Otherwise, fix $\lambda \in \mathbb{R}$ such that $0 < \lambda < |T_\epsilon(f)(x)|$. Then set $B_1 \equiv \{z \in B\colon\ |T(f)(z)| > \lambda/3\}$, $B_2 \equiv \{z \in B\colon\ |T(f_1)(z)| > \lambda/3\}$ and $B_3 \equiv \emptyset$ if $v_a\|f\|_{L^\infty(\gamma)} \leqslant \lambda/3$ and $B_3 \equiv B$ if $v_a\|f\|_{L^\infty(\gamma)} > \lambda/3$. Notice that $B = B_1 \cup B_2 \cup B_3$ and $\gamma(B) \leqslant \gamma(B_1) + \gamma(B_2) + \gamma(B_3)$. From $B \in \mathcal{B}_a$, it follows that

$$\gamma(B_1) \leqslant \frac{3}{\lambda} \int\limits_B \left|T(f)(z)\right| d\gamma(z) \leqslant \frac{3\gamma(B)}{\lambda} \mathcal{M}_a\big(T(f)\big)(x).$$

By Hölder's inequality, the fact that T is bounded on $L^2(\gamma)$ and (2.8), we obtain

$$\begin{aligned}\gamma(B_2) &\lesssim \frac{1}{\lambda} \int\limits_B \left|T(f_1)(z)\right| d\gamma(z) \lesssim \frac{[\gamma(B)]^{1/2}}{\lambda} \left\{\int\limits_B \left|T(f_1)(z)\right|^2 d\gamma(z)\right\}^{1/2} \\ &\lesssim \frac{[\gamma(B)]^{1/2}}{\lambda} \|T\|_{L^2(\gamma)\to L^2(\gamma)} \|f_1\|_{L^2(\gamma)} \\ &\lesssim \frac{[\sigma^*_{a,2}]^{1/2}\gamma(B)}{\lambda} \|T\|_{L^2(\gamma)\to L^2(\gamma)} \|f\|_{L^\infty(\gamma)}.\end{aligned}$$

Notice that if $B_3 = B$, then $\lambda < 3v_a\|f\|_{L^\infty(\gamma)}$. If $B_3 = \emptyset$, then

$$\gamma(B) \leqslant \gamma(B_1) + \gamma(B_2) \lesssim \frac{\gamma(B)}{\lambda}\left\{\mathcal{M}_a\big(T(f)\big)(x) + \sigma^*_{a,2}\|T\|_{L^2(\gamma)\to L^2(\gamma)}\|f\|_{L^\infty(\gamma)}\right\}.$$

Hence, in all cases, $\lambda \lesssim \mathcal{M}_a(T(f))(x) + \|f\|_{L^\infty(\gamma)}$. Letting $\lambda \in (0, |T_\epsilon(f)(x)|)$ and $\lambda \to |T_\epsilon(f)(x)|$ yields (4.6) with $\nu = 1$.

When $\nu \in (0,1)$, by (4.8), we know that for all $z \in B$,

$$\left|T_\epsilon(f)(x)\right|^\nu \lesssim \|f\|^\nu_{L^\infty(\gamma)} + \left|T(f)(z)\right|^\nu + \left|T(f_1)(z)\right|^\nu.$$

Taking integration average in z over B and raising to the power $1/\nu$ yield

$$\begin{aligned}\left|T_\epsilon(f)(x)\right| &\lesssim \left\{\|f\|^\nu_{L^\infty(\gamma)} + \frac{1}{\gamma(B)}\int\limits_B \left|T(f)(z)\right|^\nu d\gamma(z) + \frac{1}{\gamma(B)}\int\limits_B \left|T(f_1)(z)\right|^\nu d\gamma(z)\right\}^{1/\nu} \\ &\lesssim \|f\|_{L^\infty(\gamma)} + \left[\mathcal{M}_a\big(\left[T(f)\right]^\nu\big)(x)\right]^{1/\nu} + \left[\frac{1}{\gamma(B)}\int\limits_B \left|T(f_1)(z)\right|^\nu d\gamma(z)\right]^{1/\nu}.\end{aligned}$$

By Hölder's inequality, the fact that T is bounded on $L^2(\gamma)$ and (2.8), we have

$$\begin{aligned}\left[\frac{1}{\gamma(B)}\int_B \left|T(f_1)(z)\right|^\nu d\gamma(z)\right]^{1/\nu} &\leqslant \left[\frac{1}{\gamma(B)}\int_B \left|T(f_1)(z)\right|^2 d\gamma(z)\right]^{1/2}\\ &\leqslant \|T\|_{L^2(\gamma)\to L^2(\gamma)}\left[\frac{1}{\gamma(B)}\int_{\mathbb{R}^n} \left|f_1(z)\right|^2 d\gamma(z)\right]^{1/2}\\ &\leqslant \left[\sigma^*_{a,2}\right]^{1/2}\|T\|_{L^2(\gamma)\to L^2(\gamma)}\|f\|_{L^\infty(\gamma)}.\end{aligned}$$

Combining the last two formulae above gives (4.6) for the case $\nu \in (0,1)$. This finishes the proof of Lemma 4.1. □

Now we conclude this section with the proof of Theorem 4.1.

Proof of Theorem 4.1. For any $f \in L^\infty(\gamma)$, applying Lemma 4.1 and the fact $\gamma(\mathbb{R}^n) = 1$, we obtain

$$\left\|T^*_a(f)\right\|_{L^1(\gamma)} \lesssim \int_{\mathbb{R}^n} \left[\mathcal{M}_a\left(\left[T(f)\right]^\nu\right)(x)\right]^{1/\nu} d\gamma(x) + \|f\|_{L^\infty(\gamma)}, \tag{4.9}$$

where $\nu \in (0,1)$. Then using the facts that $\mathcal{M}_a$ is bounded on $L^{1/\nu}(\gamma)$ and that T is bounded on $L^2(\gamma)$ together with Hölder's inequality and $\gamma(\mathbb{R}^n) = 1$, we further obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \left[\mathcal{M}_a\left(\left[T(f)\right]^\nu\right)(x)\right]^{1/\nu} d\gamma(x) &\lesssim \int_{\mathbb{R}^n} \left|T(f)(x)\right| d\gamma(x)\\ &\lesssim \left\{\int_{\mathbb{R}^n} \left|T(f)(x)\right|^2 d\gamma(x)\right\}^{1/2} \lesssim \|f\|_{L^2(\gamma)} \lesssim \|f\|_{L^\infty(\gamma)}.\end{aligned}$$

Inserting this into (4.9) yields that $\|T^*_a(f)\|_{L^1(\gamma)} \lesssim \|f\|_{L^\infty(\gamma)}$. Therefore, to prove Theorem 4.1, it is enough to show that for all $f \in L^\infty(\gamma)$,

$$\sup_{B\in\mathcal{B}_a}\left[\frac{1}{\gamma(B)}\int_B T^*_a(f)(y)\,d\gamma(y) - \operatorname*{essinf}_{x\in B} T^*_a(f)(x)\right] \lesssim \|f\|_{L^\infty(\gamma)}.$$

To this end, by Proposition 3.1, we only need to prove that for all $f \in L^\infty(\gamma)$,

$$\left\|\mathfrak{M}_a T^*_a(f) - T^*_a(f)\right\|_{L^\infty(\gamma)} \lesssim \|f\|_{L^\infty(\gamma)}. \tag{4.10}$$

Now we show (4.10). Notice that $T^*_a(f)(x) \leqslant \mathfrak{M}_a T^*_a(f)(x)$ for all $f \in L^\infty(\gamma)$ and almost every $x \in \mathbb{R}^n$, and hence

$$0 \leqslant \mathfrak{M}_a T^*_a(f)(x) - T^*_a(f)(x) = \sup_{B\in\mathcal{B}_a(x)} \frac{1}{\gamma(B)}\int_B T^*_a(f)(z)\,d\gamma(z) - T^*_a(f)(x). \tag{4.11}$$

Fix any $x \in \mathbb{R}^n$ and $B \in \mathcal{B}_a(x)$. Set $f_1 \equiv f\chi_{2B}$ and $f_2 \equiv f - f_1$. Observe that

$$\begin{aligned}
&\frac{1}{\gamma(B)}\int_B T_a^*(f)(z)\,d\gamma(z)-T_a^*(f)(x)\\
&\quad\leqslant \frac{1}{\gamma(B)}\int_B T_a^*(f_1)(z)\,d\gamma(z)\\
&\qquad+\frac{1}{\gamma(B)}\int_B \big[T_a^*(f_2)(z)-T_a^*(f_2)(x)\big]\,d\gamma(z)+\big[T_a^*(f_2)(x)-T_a^*(f)(x)\big]\\
&\quad\equiv \mathrm{Z}_1+\mathrm{Z}_2+\mathrm{Z}_3.
\end{aligned}\tag{4.12}$$

Applying Lemma 4.1 for some $\nu\in(0,1)$, then using the facts that $\mathcal{M}_a$ is bounded on $L^{1/\nu}(\gamma)$ and that T is bounded on $L^2(\gamma)$ together with Hölder's inequality and (2.8), we obtain

$$\mathrm{Z}_1\leqslant \frac{1}{\gamma(B)}\int_B \big(\big[\mathcal{M}_a\big(\big[T(f_1)\big]^\nu\big)(x)\big]^{1/\nu}+\|f_1\|_{L^\infty(\gamma)}\big)\,d\gamma(z)\lesssim \|f\|_{L^\infty(\gamma)}.$$

To estimate Z_2, notice that for all $z\in B\in\mathcal{B}_a(x)$, by Lemma 2.1(i), we have $m(z)\leqslant (a+1)m(c_B)\leqslant (a+1)^2m(x)$. From this, it follows that

$$\begin{aligned}
T_a^*(f_2)(z)-T_a^*(f_2)(x)&=\sup_{0<\epsilon\leqslant am(z)}\ \inf_{0<\widetilde{\epsilon}\leqslant am(x)}\big(\big|T_\epsilon(f_2)(z)\big|-\big|T_{\widetilde{\epsilon}}(f_2)(x)\big|\big)\\
&\leqslant \max\{\mathrm{J}_1,\mathrm{J}_2\},
\end{aligned}\tag{4.13}$$

where $\mathrm{J}_1\equiv \sup_{0<\epsilon\leqslant am(x)}\inf_{0<\widetilde{\epsilon}\leqslant am(x)}|T_\epsilon(f_2)(z)-T_{\widetilde{\epsilon}}(f_2)(x)|$ and

$$\mathrm{J}_2\equiv \sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\ \inf_{0<\widetilde{\epsilon}\leqslant am(x)}\big|T_\epsilon(f_2)(z)-T_{\widetilde{\epsilon}}(f_2)(x)\big|.$$

Applying (4.5) to J_1 yields

$$\begin{aligned}
\mathrm{J}_1&\leqslant \sup_{0<\epsilon\leqslant am(x)}\big|T_\epsilon(f_2)(z)-T_\epsilon(f_2)(x)\big|\\
&\leqslant \sup_{0<\epsilon\leqslant am(x)}\int_{\mathbb{R}^n}\big|K(z,y)-K(x,y)\big|\big|f_2(y)\big|\,d\gamma(y)\leqslant v_a\|f\|_{L^\infty(\gamma)}.
\end{aligned}\tag{4.14}$$

To estimate J_2, we notice that

$$\begin{aligned}
\mathrm{J}_2&\leqslant \sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\big|T_\epsilon(f_2)(z)-T_{am(x)}(f_2)(x)\big|\\
&\leqslant \sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\big|T_\epsilon(f_2)(z)-T_\epsilon(f_2)(x)\big|\\
&\quad+\sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\big|T_\epsilon(f_2)(x)-T_{am(x)}(f_2)(x)\big|\equiv \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}\tag{4.15}$$

A similar argument to that used in the estimate of J_1 yields that $\mathrm{I}_1\leqslant v_a\|f\|_{L^\infty(\gamma)}$. To estimate I_2, using (4.4), we obtain

$$\begin{aligned}
\mathrm{I}_2&\leqslant \sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\Bigg|\int_{am(x)<|x-w|\leqslant\epsilon} K(x,w)f_2(w)\,d\gamma(w)\Bigg|\\
&\leqslant \sup_{am(x)<\epsilon\leqslant a(a+1)^2m(x)}\int_{am(x)<|x-w|\leqslant\epsilon} e^{|w|^2}\frac{C_a}{|x-w|^n}\big|f_2(w)\big|\,d\gamma(w)\lesssim \|f\|_{L^\infty(\gamma)}.
\end{aligned}\tag{4.16}$$

Combining (4.13) through (4.16) yields $\mathrm{Z}_2 \lesssim \|f\|_{L^\infty(\gamma)}$.

Finally, we estimate Z_3. If $r_B \leqslant am(x)$, write

$$T_a^*(f_2)(x) = \max\Bigl\{\sup_{0<\epsilon<r_B} \bigl|T_\epsilon(f_2)(x)\bigr|, \sup_{r_B\leqslant\epsilon\leqslant am(x)} \bigl|T_\epsilon(f_2)(x)\bigr|\Bigr\}.$$

By the support condition of f_2,

$$\sup_{0<\epsilon<r_B} \bigl|T_\epsilon(f_2)(x)\bigr| = \Biggl|\int\limits_{|x-y|>r_B} K(x,y) f_2(y)\,d\gamma(y)\Biggr| = \bigl|T_{r_B}(f_2)(x)\bigr|.$$

Therefore, we have

$$T_a^*(f_2)(x) = \sup_{r_B\leqslant\epsilon\leqslant am(x)} \bigl|T_\epsilon(f_2)(x)\bigr| \leqslant T_a^*(f)(x) + \sup_{r_B\leqslant\epsilon\leqslant am(x)} \bigl|T_\epsilon(f_1)(x)\bigr|. \tag{4.17}$$

Moreover, for all $B \in \mathcal{B}_a(x)$ and $y \in 2B$, we have

$$|x-y| \leqslant 3r_B \leqslant 3am(c_B) \leqslant 3a(a+1)m(x)$$

by Lemma 2.1(i). This combined with (4.4) further implies that when $r_B \leqslant \epsilon \leqslant am(x)$,

$$\begin{aligned}\bigl|T_\epsilon(f_1)(x)\bigr| &\leqslant \int\limits_{\{|x-y|>\epsilon,\ y\in 2B\}} \bigl|K(x,y)f(y)\bigr|\,d\gamma(y)\\ &\leqslant \int\limits_{\{|x-y|>\epsilon,\ y\in 2B\}} e^{|y|^2}\frac{C_a}{|x-y|^n}\bigl|f(y)\bigr|\,d\gamma(y) \lesssim \|f\|_{L^\infty(\gamma)}.\end{aligned} \tag{4.18}$$

Inserting this into (4.17) leads to $|\mathrm{Z}_3| \lesssim \|f\|_{L^\infty(\gamma)}$ when $r_B \leqslant am(x)$.

If $r_B > am(x)$, by the support condition of f_2, we then have

$$T_a^*(f_2)(x) = \sup_{0<\epsilon\leqslant am(x)} \bigl|T_\epsilon(f_2)(x)\bigr| = \bigl|T_{am(x)}(f_2)(x)\bigr| \leqslant T_a^*(f)(x) + \bigl|T_{am(x)}(f_1)(x)\bigr|.$$

Notice that $r_B \leqslant am(c_B) \leqslant a(a+1)m(x)$. Similarly to the estimate for (4.18), we have $|T_{am(x)}(f_1)(x)| \lesssim \|f\|_{L^\infty(\gamma)}$. Thus, $|\mathrm{Z}_3| \lesssim \|f\|_{L^\infty(\gamma)}$ when $r_B > am(x)$.

Combining the estimates of Z_1 through Z_3 with (4.12) and (4.11) then yields (4.10), which completes the proof of Theorem 4.1. □

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